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FIGURES (Cont'd)

ABSTRACT

The optimum thrust equations for two thrust programs are presented. In the first program, the thrust vector is unconstrained; in the second program, the thrust magnitude is constrained to be a constant or zero along the trajectory. These programs are obtained by a consideration of a more general thrust program containing both modes of thrust operation. The optimization process is carried out for three separate criteria: maximum final vehicle mass for a given powerplant, minimum flight time, and a minimum value of the quantity $\int_0^{t_1} a^2\, dt$, where t_1 is flight time and a is thrust acceleration. This integral is a measure of the propellant requirement for a power-limited vehicle.

These two sets of equations have been applied to an inverse square central force field model. The problem of terminal conditions is discussed and the transversality relations for both flyby and rendezvous planetary missions are developed in three dimensions. For purposes of comparison, the analytic solutions for these thrust programs in a one-dimensional field-free model are presented.

An iterative routine to solve the two-point boundary value problem has been coupled with these equations to obtain numerical solutions for specified end conditions and transversality expressions. A set of two-dimensional trajectories from Earth to Mars is presented using these two thrust programs and the various optimization criteria. A summary of the effects of these two programs on vehicle performance is presented.

CONTENTS

FIGURES

I. INTRODUCTION

For the past few years, mission-feasibility studies and trajectory analyses have been conducted to assess the payload capabilities of power-limited advanced-propulsion vehicles for various interplanetary missions. This Report describes two types of optimum thrust programs for power-limited propulsion systems which are currently being used in these studies.

The power-limited propulsion system is constrained in the amount of kinetic power contained in the exhaust propellant. A rocket equation suitable for such a system is given by

$$\frac{1}{m_1} = \frac{1}{m_0} + \int_0^{t_1} \frac{a^2}{2P}\, dt \tag{1}$$

where m_0 and m_1 are the vehicle masses at the beginning and end, respectively, of the flight, a is the thrust acceleration, and P is the power expended in the rocket exhaust. The exhaust power is determined by the power rating of the powerplant carried by the vehicle and by the efficiency of conversion by the propulsion system, which is generally dependent on the exhaust velocity. The final vehicle mass depends on the value of this integral which, in turn, depends upon the flight time, the mission involved (namely, the specification of the kinematic conditions of the vehicle initially and terminally), the force field in which the vehicle travels, the nature of the thrust program used to accomplish this mission, and, finally, the engineering design of the propulsion system.

For preliminary mission-feasibility studies, it is desirable to employ thrust programs which exclude the complexity imposed by the engineering design but which bracket or isolate that class of trajectories and vehicle performances which an actual vehicle should be capable of achieving. Two such thrust programs which serve this purpose are presented here. The first program allows a freely varying thrust magnitude and direction. The second program constrains the thrust magnitude to some constant value or zero but allows a freely varying thrust direction.

These thrust programs have been optimized with respect to certain criteria. The variable thrust program has been optimized so that, for a particular mission,

$$\int_0^{t_1} a^2\, dt = \text{minimum}$$

This thrust program yields the absolute minimum that $\int_0^{t_1} a^2\, dt$ may have and gives rise to the so-called optimum thrust equations of power-limited flight (1, 2). Its justification stems from the fact that, over a wide range of exhaust velocity, the exhaust power is nearly constant; although, this is generally not the case for low exhaust velocity.

The constant thrust program has been optimized with respect to three separate criteria. For comparison with the variable thrust program, trajectories with minimum $\int_0^{t_1} a^2\, dt$ are generated. The constant thrust program is constrained to constant thrust (and hence, constant exhaust velocity) or coast periods but minimizes this integral over the propulsion periods by optimum programming of the thrust vector and by optimum selection of the burning periods. The resulting value is always higher than the first case and, therefore, yields more conservative estimates of final vehicle mass. For a particular mission, then, the generation of a pair of trajectories and vehicle performances using these two thrust programs is extremely valuable in determining mission feasibility, payload capability, trajectory design, et cetera. The second criterion yields trajectories with maximum final vehicle mass when a particular propulsion system is used by optimizing the constant exhaust velocity. The third criterion yields *minimum-time* trajectories. The differences among these types of trajectories will become clearer in the subsequent discussion.

II. THRUST OPTIMIZATION

The optimization of these thrust programs may be accomplished by calculus of variations methods in which the desired quantities are extremized subject to certain boundary conditions and certain constraints; namely, the equations of motion, the thrust-program constraints, et cetera. The constraining equations for this problem are

$$\dot{\mathbf{v}} + \nabla U - \mathbf{a} = \mathbf{0} \tag{2}$$

$$\mathbf{v} - \dot{\mathbf{r}} = \mathbf{0} \tag{3}$$

$$a - \frac{\beta}{c} \frac{\alpha_p}{\mu} = 0 \tag{4}$$

$$\dot{\mu} + \frac{\beta}{c^2} \alpha_p = 0, \qquad \mu(0) = 1 \tag{5}$$

Equations (2) and (3) are the rocket equations of motion, where $\mathbf{r}$ is the position vector of the vehicle and U is the potential of the force field. The thrust acceleration is controlled by Equation 4, in which μ is the normalized vehicle mass and α_p is a normalized power parameter ranging between a value of 1 (maximum power) and 0; it will be shown subsequently to have a value of 1 during propulsion periods and 0 during coasting periods. The bounds on α_p may be expressed in analytic form through the constraining relation

$$\gamma^2 - \alpha_p(1 - \alpha_p) = 0 \tag{6}$$

where γ is defined to be a real variable. The quantity β is $2P_{max}/m_0$ and is, therefore, a constant determined by the engineering design. The quantity c is the rocket-exhaust velocity which is either continuously variable or constant, depending on the thrust program used. Equation (5) is the normalized differential form of Equation (1).

Both the variable and the constant thrust programs may be solved by a consideration of a more general thrust program containing both of these modes, in which c is allowed to vary between two bounds. As before, these bounds may be stated through the expression

$$\eta^2 - (c_{max} - c)(c - c_{min}) = 0 \tag{7}$$

where η is defined to be a real variable. By solving the calculus of variations problem with this additional constraint, the optimum thrust equations for the programs under discussion are obtained very simply by setting $c_{min} = 0$ and $c_{max} = \infty$ for the first case and by setting $c_{min} = c_{max}$ = constant for the second case.

A Mayer formulation (3) has been applied to this more general problem to obtain the optimum thrust equations. The present treatment is similar to that followed by Miele (4), Lawden (5), Leitmann (6), and others.

Let $q_j(t)$ denote both the state and the control variables of the problem ($j = 1, 2, \cdots, n$). Let the constraining relations be denoted by the functions

$$G_i(q_j, \dot{q}_j, t) = 0, \qquad i = 1, 2, \cdots, m < n \tag{8}$$

and let $\lambda_i(t)$ be a set of time-dependent Lagrange multipliers. Let F be a function defined by

$$F = \lambda_i \, G_i \tag{9}$$

where the summation rule is employed. The quantity to be extremized is given by $J\,[q_j(t_1), q_j(0), t_1]$, that is, a function of the variables at the end points only. As necessary conditions for extremizing J, the q_j's must satisfy the Euler-Lagrange equations given by

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) - \frac{\partial F}{\partial q_j} = 0, \qquad j = 1, 2, \cdots, n \tag{10}$$

at all points along the trajectory except at corners; that is, points of discontinuity in one or more of the q_j's. Further, at such corners the Weierstrass-Erdmann corner conditions must hold; namely,

$$\frac{\partial F}{\partial \dot{q}_j} \text{ is continuous} \qquad j = 1, 2, \cdots \tag{11}$$

$$F - \frac{\partial F}{\partial \dot{q}_j} \dot{q}_j \text{ is continuous} \tag{12}$$

If the constraining functions are not explicit functions of time, a first integral of the Euler-Lagrange equation is

$$F - \frac{\partial F}{\partial \dot{q}_j} \dot{q}_j = \text{constant} \tag{13}$$

One additional tool from the calculus of variations will be needed in dealing with corners; namely, the Weierstrass E-function. This function yields a further necessary condition for the minimization of J by the inequality

$$E = F(q_1^*, \cdots, q_n^*, \dot{q}_1^*, \cdots, \dot{q}_n^*) - F(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) - (\dot{q}_j^* - \dot{q}_j) \frac{\partial F}{\partial \dot{q}_j} \geq 0 \tag{14}$$

The q_j^* is an admissible value in the vicinity of q_j. For continuous variables, $q_j^* = q_j$; however, for discontinuous variables, q_j^* may take on any value consistent with the specified bounds.

In three dimensions, Equations (2) through (7) form a system of ten constraining relations which must be included in the optimization process. There are, in this formulation, fourteen variables, of which $\mathbf{v}$, $\mathbf{r}$, and μ are the state variables; a_p, c, and $\mathbf{a}$ are the control variables with the latter being related through

Equation (4); the quantities η and γ are auxiliary variables. In this problem, G_1, G_2, and G_3 are the equations of motion contained in Equation (2); the functions G_7, G_8, G_9, and G_{10} are Equations (4), (5), (6), and (7), respectively.

The Euler-Lagrange equations for this problem are given by the relations

$$\mathbf{v}, \mathbf{r}: \qquad \ddot{\boldsymbol{\lambda}} + (\boldsymbol{\lambda} \cdot \nabla)\, \nabla U = \mathbf{0} \tag{15}$$

$$\mathbf{a}: \qquad \mathbf{a} - \left(\frac{a}{\lambda_7}\right) \boldsymbol{\lambda} = \mathbf{0} \tag{16}$$

$$\mu: \qquad \dot{\lambda}_8 - \left(\frac{\beta \alpha_p}{c\,\mu^2}\right) \lambda_7 = 0 \tag{17}$$

$$c: \qquad \frac{\beta}{c^2}\, \alpha_p \left(\frac{\lambda_7}{\mu} - \frac{2\lambda_8}{c}\right) + \lambda_{10}(2c - c_{min} - c_{max}) = 0 \tag{18}$$

$$\alpha_p: \qquad \frac{\beta}{c}\left(\frac{\lambda_7}{\mu} - \frac{\lambda_8}{c}\right) - \lambda_9(2\alpha_p - 1) = 0 \tag{19}$$

$$\gamma: \qquad \gamma \lambda_9 = 0 \tag{20}$$

$$\eta: \qquad \eta \lambda_{10} = 0 \tag{21}$$

where λ_4, λ_5, and λ_6 have been eliminated. The quantity $\boldsymbol{\lambda}$ is the vector sum of the three orthogonal quantities λ_1, λ_2, and λ_3; thus,

$$\lambda = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \pm\lambda_7 \tag{22}$$

where the latter relation follows directly from Equation (16). Since U is assumed to be explicitly independent of time, these equations admit a first integral in the form

$$\dot{\boldsymbol{\lambda}} \cdot \mathbf{v} + \boldsymbol{\lambda} \cdot \nabla U - \frac{\beta \alpha_p}{c}\left(\frac{\lambda_7}{\mu} - \frac{\lambda_8}{c}\right) = \text{constant} = K_2 \tag{23}$$

An application of the Weierstrass-Erdmann corner conditions yields the following summary:

1. Continuous variables: $\mathbf{r}$, $\mathbf{v}$, μ, $\boldsymbol{\lambda}$, $\dot{\boldsymbol{\lambda}}$, λ_8, and K_2
2. Possibly discontinuous variables: $\mathbf{a}$, $\dot{\mu}$, c, η, α_p, γ, λ_7, $\dot{\lambda}_8$, λ_9, and λ_{10}

In view of the above continuity considerations, the Weierstrass E-function becomes

$$E = \frac{\alpha_p}{c}\left(\frac{\lambda_7}{\mu} - \frac{\lambda_8}{c}\right) - \frac{\alpha_p^*}{c^*}\left(\frac{\lambda_7^*}{\mu} - \frac{\lambda_8}{c^*}\right) \geq 0 \tag{24}$$

Since this condition holds for all admissible values of the pertinent variables, it holds, in particular, when $\alpha_p = \alpha_p^*$ and $c = c^*$. From this, the sign ambiguity appearing in Equation (22) is resolved and the positive sign is taken in order to satisfy Equation (24). The quantity λ_7 is simply replaced by λ. The function $L(t)$, defined

$$L = \frac{\lambda}{\mu} - \frac{\lambda_8}{c} \tag{25}$$

is substituted in Equation (24) to yield

$$\frac{\alpha_p L}{c} - \frac{\alpha_p^* L^*}{c^*} \geq 0 \tag{26}$$

In this treatment, c is restricted to be either a continuous variable or a constant, as indicated in Equation (7). (The case of a discretely varying exhaust velocity is considered in the Appendix.) In this case, $L(t)$ is a continuous function and Equation (26) reduces to

$$L(\alpha_p - \alpha_p^*) \geq 0 \tag{27}$$

It will be shown subsequently that α_p is restricted to the values of 1 and 0. It follows from Equation (27) that

$$\begin{aligned} &1.\quad L > 0, \qquad \alpha_p = 1 \\ &2.\quad L < 0, \qquad \alpha_p = 0 \end{aligned} \tag{28}$$

thus, negative values of $L(t)$ indicate coasting periods along the trajectory. Furthermore, the continuity of K_2 implies that α_p may change in value only at points where $L(t)$ is zero.

Equation (20) implies that either λ_9 is zero (α_p variable) or γ is zero ($\alpha_p = 1, 0$). If λ_9 is zero, Equation (19) implies

$$\lambda_8 = c\,\frac{\lambda}{\mu} \tag{29}$$

which, when combined with Equations (5) and (17), yields

$$\lambda_8 = \frac{\text{constant}}{\mu} \tag{30}$$

However, Equation (19) then implies that

$$\lambda = \frac{\text{constant}}{c} \tag{31}$$

Now, Equation (21) shows that λ_{10} is zero in the variable thrust mode. When λ_{10} is zero, Equation (18) is incompatible with Equation (31), from which it follows that Equation (31) can hold only during the constant thrust mode and therefore

$$\lambda = \text{constant} \tag{32}$$

The occurrence of a constant λ is extremely unlikely in most potential fields. The two- and three-dimensional harmonic oscillator potential is a noticeable exception. Lawden (7) has recently discussed this singular case for a two-dimensional inverse square field. Corben (8) has shown for this case that the direction of $\boldsymbol{\lambda}$ relative to the local horizontal is constrained to lie within approximately 35 deg of the local horizontal. The combination of these two constraints of constant λ and bounded direction make Equation (32) inadmissable for the planetary rendezvous and flyby missions to be discussed subsequently. For this treatment, then, Equation (32) is considered inadmissable, and it follows that λ_9 [and therefore, $L(t)$] is not zero except at discrete points along the trajectory (and is, in fact, determined by Equation 19) and therefore, α_p is either 1 or zero, depending on the sign of $L(t)$.

Consider now the transfer from one thrust mode to the other. If λ_{10} is zero, it follows from Equation (18) that

$$\frac{2\beta}{c^2}\,\alpha_p\left[L - \frac{\lambda}{2\mu}\right] = 0 \tag{33}$$

and since $\lambda/2\mu$ is positive, the non-trivial conclusion is that $L(t)$ is positive and that, during the variable thrust mode (VTM),

$$L - \frac{\lambda}{2\mu} = 0 \qquad \text{(VTM)} \tag{34}$$

The quantity α_p is equal to 1 during the variable thrust mode.

$$\lambda_8 = \frac{A}{\mu^2} \qquad \text{(VTM)} \tag{35}$$

which, in turn, yields for the exhaust velocity

$$c = \frac{2A}{\mu\lambda} \qquad \text{(VTM)} \tag{36}$$

and for the thrust acceleration

$$a = \frac{\beta\lambda}{2A} \qquad \text{(VTM)} \tag{37}$$

where A is a constant determined by initial conditions. If λ_{10} is not zero, η must be zero and the constant thrust mode (CTM) is operative. In this case, $L(t)$ is unrestricted and is given by Equation (25). In numerical studies, it has been convenient to eliminate λ_8 because of its dependence on exhaust velocity. In the limiting case of infinite exhaust velocity (constant thrust acceleration), both Equations (17) and (25) encounter difficulty. This is obviated by employing a differential equation in $L(t)$ in place of Equation (17). It is easily shown, for the constant thrust mode using Equations (5), (17), and (25), that

$$\dot{L} - \frac{\dot{\lambda}}{\mu} = 0 \qquad \text{(CTM)} \tag{38}$$

In summary, the continuity of K_2 requires that the transfer from one thrust mode to another occurs when the conditions

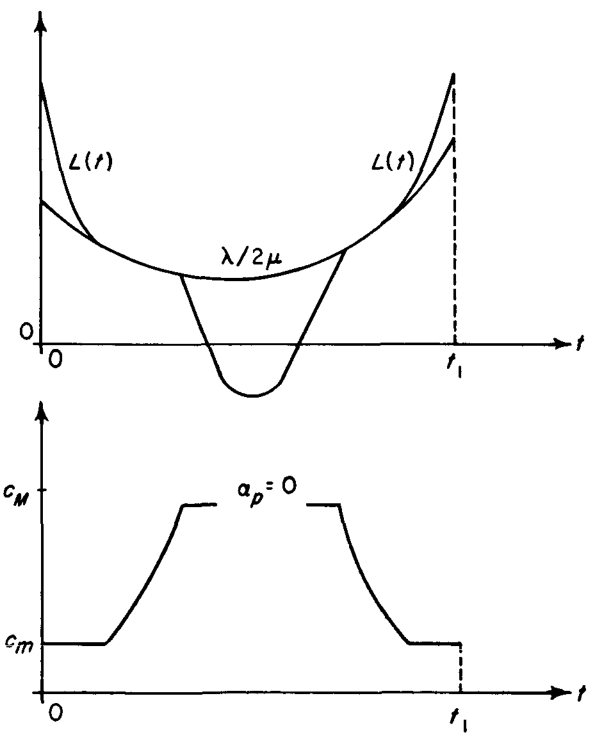
$$\begin{aligned} &1.\quad L = \frac{\lambda}{2\mu} \\ &2.\quad \eta = 0 \end{aligned} \tag{39}$$

occur simultaneously. A typical sequence is shown in the accompanying sketches (Sketch 1). The thrust acceleration in the CTM is given by

$$a = \frac{\beta \alpha_p}{c \mu} \frac{\boldsymbol{\lambda}}{\lambda}, \qquad c = c_{min} \text{ or } c_{max} \tag{40}$$

and in the VTM by

$$a = \frac{\beta \boldsymbol{\lambda}}{2A} \tag{41}$$



Sketch 1

III. OPTIMIZATION CRITERIA

As stated in the Introduction, the three criteria to be considered for the constant thrust program are (1) maximum final vehicle mass, (2) minimum $\int_0^{t_1} a^2\, dt$, and (3) minimum flight time for a given final vehicle mass. The variable thrust program satisfies all three of these criteria simultaneously, if the exhaust power is constant. The constant thrust program, however, possesses coast periods, and by allocating different-length coast periods, one can obtain trajectories satisfying different criteria.

From the calculus of variations, one has certain transversality conditions which hold at end points of the trajectory. There may be boundary conditions which may be formulated in terms of the variables of the problem. Let these conditions be separable into initial and terminal conditions described by the functions

$$A_\nu(q_i, t)\Big|_{t=0} = 0, \qquad \nu = 1, 2, \cdots, \leq n+1 \tag{42}$$

and

$$B_\nu(q_i, t)\Big|_{t=t_1} = 0, \qquad \nu = 1, 2, \cdots, \leq n+1 \tag{43}$$

It may be shown that the transversality conditions for this formulation are

$$\left[\left(\dot{q}_i \frac{\partial A_\nu}{\partial q_i} \kappa_\nu + \frac{\partial A_\nu}{\partial t} \kappa_\nu + F - \dot{q}_i \frac{\partial F}{\partial \dot{q}_i}\right) \delta t + \left(\frac{\partial A_\nu}{\partial q_i} \kappa_\nu + \frac{\partial F}{\partial \dot{q}_i}\right) \delta q_i\right]\Bigg|_{t=0} = 0 \tag{44}$$

and

$$\left[\left(\dot{q}_i \frac{\partial B_\nu}{\partial q_i} \kappa_\nu + \frac{\partial B_\nu}{\partial t} \rho_\nu + F - \dot{q}_i \frac{\partial F}{\partial \dot{q}_i}\right) \delta t + \left(\frac{\partial B_\nu}{\partial q_i} \rho_\nu + \frac{\partial F}{\partial \dot{q}_i}\right) \delta q_i\right]\Bigg|_{t=t_1} = 0 \tag{45}$$

where the κ_ν and ρ_ν are Lagrange multiplier constants to be determined by simultaneous solution of Equations (42) through (45). The δ quantities are arbitrary variations in the variables at the terminal point. If the value of a variable is specified at an end point, the corresponding variation is zero.

This is now applied to the criterion of maximum final mass for fixed terminal time and end conditions. At a fixed terminal time, let

$$B_1 = \mu(t_1) - \mu_{max} \tag{46}$$

and let all the remaining end conditions be independent of μ. An application of Equation (45) yields

$$\lambda_8(0) = \text{unspecified} \tag{47}$$

$$\lambda_8(t_1) = -\rho_1 \tag{48}$$

However, λ_8 is a monotonic increasing function and, since ρ_1 is undetermined, it may be any negative constant. (This is consistent with the fact that $\rho_1 \mu$ should be minimized in order to satisfy the Weierstrass E-function condition. In fact, the adoption of the positive sign in Equation (22) dictates a positive value for λ_8.) It is observed that Equations (15) through (23) are homogeneous in the Lagrange multipliers; they may, therefore, be scaled without affecting the trajectory. Consequently, it is quite unnecessary, as some writers have done, to relate a particular Lagrange multiplier to some variable such as μ. For a particular choice of values of β and c, and for specified terminal conditions, the Euler-Lagrange equations guarantee an extremal in μ.

However, for a particular β and for specified end conditions, there is an optimum choice of c which maximizes the final vehicle mass. Isolating this value of c may be accomplished by introducing a new constraining equation into the formulation, in the form

$$\dot{c} = 0 = G_{11} \lambda_{11} \frac{\beta}{c^2} \tag{49}$$

and ignoring Equation (18), since c is a constant. The Euler-Lagrange equation for this expression is

$$\dot{\lambda}_{11} - a_p \left(2L - \frac{\lambda}{\mu} \right) = 0 \tag{50}$$

and the transversality equations yield

$$\lambda_{11}(0) = \lambda_{11}(t_1) = 0 \tag{51}$$

A trajectory which terminates with a value of zero for λ_{11} therefore possesses an extremal in μ with respect to c.

Consider now the minimization of $\int_0^{t_1} a^2\, dt$. Satisfying Equation (51) also yields an extremal in this integral with respect to c, since it may be written as

$$\int_0^{t_1} a^2\, dt = \beta \left[\frac{1}{\mu(t_1)} - \frac{1}{\mu(0)} \right] \tag{52}$$

It is advantageous, however, to consider the minimization of this integral for a fixed c and fixed t_1 by changing the initial thrust acceleration a_0. This integral is fairly insensitive to the value chosen for c so long as it is in the range typical of low-thrust propulsion systems; it approaches a limiting value for infinite exhaust velocity. Dividing by c to deal also with the limiting case, an a_0 will be found which extremizes the expression

$$\frac{1}{c} \int_0^{t_1} a^2\, dt = a_0 \left(\frac{1}{\mu(t_1)} - \frac{1}{\mu(0)} \right), \qquad \mu(0) = 1 \tag{53}$$

The approach is the same as followed in Equation (49); the additional constraining relation

$$\dot{a}_0 = 0 = G_{12} \lambda_{12} \tag{54}$$

is introduced with Equation (18) again being ignored. The resulting Euler-Lagrange equation is

$$\dot{\lambda}_{12} + \alpha_p L = 0 \tag{55}$$

An application of Equation (44) yields

$$\lambda_{12}(0) = 0 \tag{56}$$

At the terminal point, both a_0 and $\mu(t_1)$ are unspecified and one has the boundary condition

$$B_1 = a_0 \left[\frac{1}{\mu(t_1)} - 1 \right] - \left(\int_0^{t_1} a^2\, dt \right) \Bigg|_{max} \tag{57}$$

Equation (45) yields the expressions

$$\lambda_8 - \frac{\rho_1 a_0}{\mu^2} = 0 \tag{58}$$

$$\lambda_{12} + \rho_1 \left(\frac{1-\mu}{\mu}\right) = 0 \tag{59}$$

which, upon eliminating ρ_1, yields the transversality expression

$$\left[\lambda_{12} + (1-\mu)\lambda_8 \frac{\mu}{a_0}\right]_{t_1} = 0 \tag{60}$$

Eliminating λ_8 and $1-\mu$, one obtains the function $R(t)$, given by

$$R(t) = \lambda_{12} + (\lambda - \mu L) \int_0^t \alpha_p \, dt \tag{61}$$

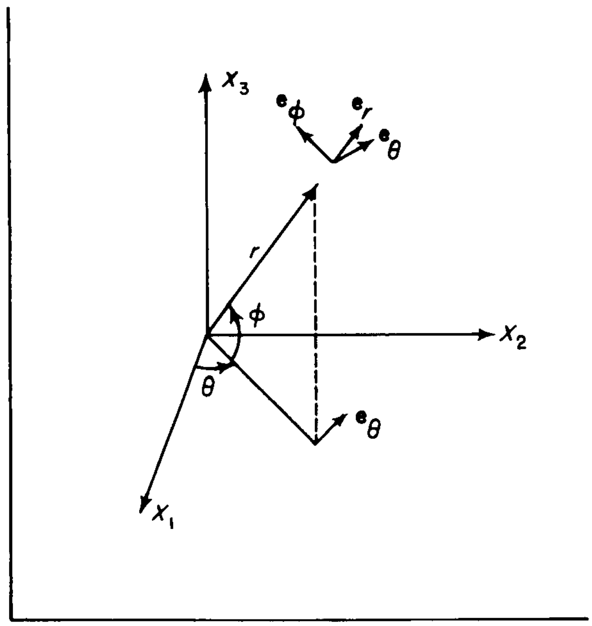
which must be zero at t_1 to guarantee an extremal value in the $\int_0^{t_1} a^2 \, dt$ with respect to the initial acceleration. It can be shown, for infinite exhaust velocity, that $R(t)$ and $-\lambda_{11}(t)$ are identical.

For minimum time trajectories with specified end conditions and specified $\mu(t_1)$ or $\int_0^{t_1} a^2 \, dt_1$, one simply forces the propulsion system to operate over the whole trajectory.

IV. THREE-DIMENSIONAL SPHERICAL COORDINATE REPRESENTATION

For computational purposes, it is advantageous to employ a coordinate system which capitalizes on the symmetry properties of the problem. The computations have been performed in a two-body inverse square force field; the potential U is given by

$$U = -\frac{GM}{r} \tag{62}$$



Sketch 2

The spherical coordinate system and the directions of the basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are shown in Sketch 2. In this representation, the Lagrange multiplier vector becomes

$$\boldsymbol{\lambda} = \mathbf{e}_r\lambda_1 + \mathbf{e}_\theta\lambda_2 + \mathbf{e}_\phi\lambda_3 \tag{63}$$

and the thrust acceleration is

$$\mathbf{a} = \mathbf{e}_r a_r + \mathbf{e}_\theta a_\theta + \mathbf{e}_\phi a_\phi \tag{64}$$

The kinematic constraining equations are

$$\ddot{r} - \frac{h^2}{r^3} + \frac{GM}{r^2} - a_r = 0 \tag{65}$$

$$\dot{h}_\theta - \frac{h_\phi^2 \tan\phi}{r^2} + r a_\phi = 0 \tag{66}$$

$$\dot{h}_\phi + \frac{h_\theta h_\phi \tan\phi}{r^2} - r a_\theta = 0 \tag{67}$$

$$h_\theta + r^2 \dot{\phi} = 0 \tag{68}$$

$$h_\phi - r^2 \dot{\theta} \cos\phi = 0 \tag{69}$$

where h is the angular momentum per unit vehicle mass and in vectorial form is

$$\mathbf{h} = \mathbf{e}_\theta h_\theta + \mathbf{e}_\phi h_\phi \tag{70}$$

After some manipulation, it may be shown that Equation (15) becomes

$$\ddot{\lambda}_1 + \left(\frac{3h^2}{r^4} - \frac{2GM}{r^3}\right)\lambda_1 - \frac{1}{r}\left(a_\theta \lambda_2 + a_\phi \lambda_3\right) - \frac{2h_\theta F(t)}{r^3} - \frac{2h_\phi (\lambda \cdot \mathbf{h})}{r^4} \tan\phi - \frac{2h_\phi K_1}{r^3 \cos\phi} = 0 \tag{71}$$

$$r^2 \frac{d}{dt}\left(\frac{\lambda_2}{r}\right) + \frac{h_\phi}{r}\left(2\lambda_1 - \lambda_3 \tan\phi\right) - (\lambda \cdot \mathbf{h}) \frac{\tan\phi}{r} - \frac{K_1}{\cos\phi} = 0 \tag{72}$$

$$r^2 \frac{d}{dt}\left(\frac{\lambda_3}{r}\right) - \frac{2h_\theta \lambda_1}{r} + \frac{h_\phi \lambda_2 \tan\phi}{r} + F(t) = 0 \tag{73}$$

$$\dot{F}(t) - \frac{h_\phi}{r^3 \cos^2 \phi}(\boldsymbol{\lambda} \cdot \mathbf{h}) - \frac{K_1 h_\phi \sin \phi}{r^2 \cos^2 \phi} = 0 \tag{74}$$

The quantity $F(t)$ is an auxiliary variable and is, essentially, one of the Lagrange multipliers. The constant, K_1, is a constant of integration resulting from the cyclic nature of the variable θ. Equation (23) becomes

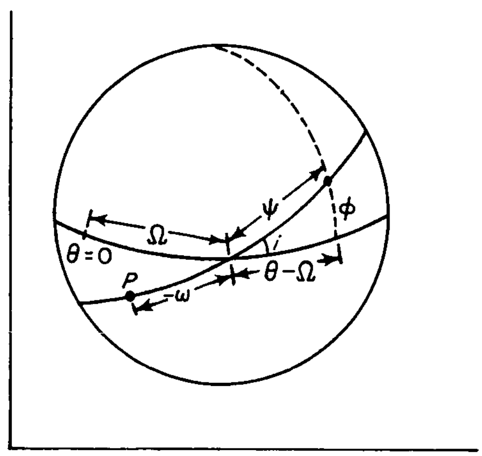
$$\dot{\lambda}_1 \dot{r} + \left(\frac{GM}{r^2} - \frac{h^2}{r^3}\right) \lambda_1 + \frac{h_\phi (\boldsymbol{\lambda} \cdot \mathbf{h})}{r^3} \tan \phi + \frac{h_\theta F(t)}{r^2} + \frac{K_1 h_\phi}{r^2 \cos \phi} - a \mu \alpha_p L = K_2 \tag{75}$$

The thrust acceleration, $\mathbf{a}$, is given by Equations (40) and (41), depending on the thrust mode employed. Similarly, $L(t)$ is obtained from either Equation (34) or (38) and the conditions for transferring thrust modes are given in Equation (39). The conditions for coasting follow from Equation (28). The vehicle mass μ is obtained from Equation (5), and in the variable thrust mode, c is found from Equation (36). Finally, the quantities λ_{11}, λ_{12}, and $R(t)$ are obtained from Equations (50), (55), and (61), respectively.

V. MISSIONS AND TERMINAL CONDITIONS

The kinematic variables in most missions are specified at the initial point of a trajectory and, in a final trajectory design, the terminal values are usually specified. In a preliminary study, however, it is advantageous to allow certain terminal variables to be free in order to optimize the trajectory with respect to certain criteria such as payload capability, communication distance, error sensitivity, et cetera. The transversality conditions which result from certain unspecified terminal variables for various types of missions are presented. These missions include planetary rendezvous, planetary flyby, and orbital inclination changes. The initial conditions, as characterized by six kinematic variables, are fixed.

In planetary-rendezvous missions, six terminal quantities must be specified. It is convenient to group these into five quantities which determine the shape and orientation of the terminal ellipse and one quantity indicating the rendezvous position on the ellipse. These quantities are the energy per unit mass, E; the angular momentum per unit mass, h; the orbital inclination, i; the argument of perigee, ω; the longitude of the ascending node, Ω; and the angle from the line of nodes to the rendezvous point, ψ (Sketch 3).



Sketch 3

These quantities are expressed in terms of the six kinematic variables through the relations

$$E = \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \frac{GM}{r} \tag{76}$$

$$h^2 = h_\theta^2 + h_\phi^2 \tag{77}$$

$$\cos i = \frac{h_\phi \cos\phi}{h}, \qquad 0 \le i \le \pi \tag{78}$$

$$\omega = \psi - \sin^{-1}\frac{h\dot{r}}{GMe}; \qquad \omega = \psi - \cos^{-1}\left[\frac{1}{e}\left(\frac{h^2}{GMr} - 1\right)\right], \qquad 0 \le \omega \le 2\pi \tag{79}$$

$$\sin\Omega = \frac{-h_\theta \sin\theta - h_\phi \sin\phi\cos\theta}{h\sin i}$$

$$\cos\Omega = \frac{-h_\theta\cos\theta + h_\phi\sin\phi\sin\theta}{h\sin i}$$

$$0 \le \Omega \le 2\pi \tag{80}$$

$$\sin\psi = \frac{\sin\phi}{\sin i}, \qquad \cos\psi = \frac{-h_\theta\cos\phi}{h\sin i} \qquad 0 \le \psi \le 2\pi \tag{81}$$

where e is the eccentricity of the ellipse. These six expressions serve as boundary conditions at the terminal point of the trajectory. For each one of these conditions which is left unspecified, there is a corresponding transversality expression as indicated by Equation (45). These transversality expressions yield extremals in the quantity to be optimized with respect to the unspecified boundary conditions. Both relative maxima and minima result from satisfying these conditions.

The tabulation below lists several useful combinations of rendezvous terminal conditions and corresponding transversality relations obtained from an application of Equation (45) for a fixed t_1.

No.	I	II	III	IV	V	VI
1	$E = E_s$	$E = E_s$	$E = E_s$	$E = E_s$	$E = E_s$	$E = E_s$
2	$h = h_s$	$h = h_s$	$h = h_s$	$h = h_s$	$h = h_s$	$h = h_s$
3	$i = i_s$	$i = i_s$	$i = i_s$	$i = i_s$	$i = i_s$	$K_1 = 0$
4	$\omega = \omega_s$	$\omega = \omega_s$	$\omega = \omega_s$	$\Omega = \Omega_s$	$K_1 = 0$	$M = 0$
5	$\Omega = \Omega_s$	$\Omega = \Omega_s$	$K_1 = 0$	$M = 0$	$M = 0$	$\boldsymbol{\lambda} \cdot \mathbf{h} = 0$
6	$\psi = \psi_s$	$M + N + \dfrac{K_1 h_\phi}{r^2 \cos \phi} = 0$	$M + N = 0$	$N + \dfrac{K_1 h_\phi}{r^2 \cos \phi} = 0$	$N = 0$	$F = 0$

The subscript, s, denotes a specified terminal value. The functions M and N are given by

$$M = \dot{\lambda}_1 \dot{r} + \lambda_1 \left(\frac{GM}{r^2} - \frac{h^2}{r^3} \right) \tag{82}$$

$$N = \frac{h_\phi (\boldsymbol{\lambda} \cdot \mathbf{h})}{r^3} \tan \phi + \frac{h_\theta F(t)}{r^2} \tag{83}$$

and will be recognized as components of Equation (75). Combination III is most useful when the trajectory commences from a circular orbit which relaxes the necessity of specifying $\Omega(t_1)$. Combinations IV and V apply to circular terminal orbits and to the case of orbital inclination changes. Combination VI applies also to the two-dimensional case where $\boldsymbol{\lambda} \cdot \mathbf{h}$ and $F(t)$ are zero over the trajectory. This particular case has been noted by Blum (1) for the variable thrust program. The quantity K_1 is zero when neither θ nor any quantity explicitly dependent on θ is specified.

For flyby trajectories, the terminal velocity vector is generally unspecified. The following tabulation lists several flyby-type terminal conditions and the corresponding transversality conditions.

No.	VII	VIII	IX	X	XI
1	$r = r_s$	$r = r_s$	$r = r_s$	$r = r_s$	$E = E_s$
2	$\theta = \theta_s$	$\theta = \theta_s$	$K_1 = 0$	$\dot{r} = \dot{r}_s$	$K_1 = 0$
3	$\phi = \phi_s$	$F = 0$	$F = 0$	$K_1 = 0$	$F = 0$
4	$\lambda_1 = 0$	$\lambda_1 = 0$	$\lambda_1 = 0$	$F = 0$	$M = 0$
5	$\lambda_2 = 0$	$\lambda_2 = 0$	$\lambda_2 = 0$	$\lambda_2 = 0$	$\boldsymbol{\lambda} \cdot \mathbf{h} = 0$
6	$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_3 = 0$	$\lambda_1 h_\phi - \lambda_2 r\dot{r} = 0$

Combination X applies to a "tangential flyby" and to such missions as a solar probe. Combination XI applies to the case of maximizing the terminal energy; the fifth and sixth conditions characterize the well-known fact that the terminal thrust vector is parallel to the velocity vector.

VI. ONE-DIMENSIONAL EXAMPLE

As a simple example of the foregoing results, the one-dimensional rendezvous and flyby missions are considered in a field-free region. For the purpose of simplicity, the mass loss of the vehicle is zero ($\mu = 1$). The boundary conditions are

$$x(0) = 0 \qquad x(t_1) = L$$

$$\dot{x}(0) = 0 \qquad \dot{x}(t_1) = 0, \text{ rendezvous}$$

The terminal velocity is unspecified in the flyby case. For this case, Equation (15) yields, after integration

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \dot{\boldsymbol{\lambda}}_t \tag{84}$$

where $\dot{\boldsymbol{\lambda}}$ is a constant. For the variable thrust program, an application of Equations (2), (3), (41), and the above boundary condition yields

$$a = \begin{cases} \dfrac{6L}{t_1^2}\left(1 - \dfrac{2t}{t_1}\right), & \text{rendezvous} \\[2ex] \dfrac{3L}{t_1^2}\left(1 - \dfrac{t}{t_1}\right), & \text{flyby} \end{cases} \tag{85}$$

where, in the case of the flyby, the condition that λ is zero at t_1 has been applied. For these two cases,

$$\int_0^{t_1} a^2\, dt = \begin{cases} \dfrac{12L^2}{t_1^3}, & \text{rendezvous} \\[2ex] \dfrac{3L^2}{t_1^3}, & \text{flyby} \end{cases} \tag{86}$$

In the constant thrust case, it is recalled that $\dot{L}$ is proportional to the time derivative of the magnitude of $\boldsymbol{\lambda}$, therefore

$$\dot{L} = \begin{cases} \dot{\lambda}, & \lambda > 0 \\ -\dot{\lambda}, & \lambda < 0 \end{cases} \tag{87}$$

from which it follows that there are, at most, two zero crossings of L. Since $\dot{x}(0)$ is zero, the trajectory commences with a burning phase and therefore has, at most, one coast period. Let t_a denote the beginning of the coast period, and for the rendezvous case let t_b denote the second burn period in which, by Equation (87), the thrust is negative. An application of Equations (2), (3), (16), and the boundary conditions yield

$$|a| = \begin{cases} \dfrac{L\,a_p}{t_a t_b}, & \text{rendezvous} \\[2ex] \dfrac{L\,a_p}{t_a\left(t_1 - \frac{1}{2}t_a\right)}, & \text{flyby} \end{cases} \tag{88}$$

and for the rendezvous case

$$t_1 = t_a + t_b \tag{89}$$

From Equation (50), one finds that the necessary values for t_a so that λ_{11} is zero at t_1 are

$$t_a = \begin{cases} \dfrac{1}{3}\,t_1, & \text{rendezvous} \\[2ex] \dfrac{2}{3}\,t_1, & \text{flyby} \end{cases} \tag{90}$$

With this value

$$\int_0^{t_1} a^2 = \begin{cases} \dfrac{27}{2}\,\dfrac{L^2}{t_1^3}, & \text{rendezvous} \\[2ex] \dfrac{27}{8}\,\dfrac{L^2}{t_1^3}, & \text{flyby} \end{cases} \tag{91}$$

which is 12.5% larger than the variable thrust program.

The acceleration level corresponding to a minimum-time trajectory is obtained from equating t_a and t_b in the rendezvous case and equating t_a and t_1 for the flyby. In this case,

$$|a| = \begin{cases} \dfrac{4L}{t_1^2}, & \text{rendezvous} \\[2ex] \dfrac{2L}{t_1^2}, & \text{flyby} \end{cases} \tag{92}$$

and for this case

$$\int_0^{t_1} a^2\, dt = \begin{cases} 16\, \dfrac{L^2}{t_1^3}, & \text{rendezvous} \\[2ex] 4\, \dfrac{L^2}{t_1^3}, & \text{flyby} \end{cases} \tag{93}$$

which is 33% larger than the variable thrust program and 18.5% higher than the value for the optimum coast trajectory. These examples, although trivial, do lend insight into the more complicated problems of interplanetary trajectories where similar characteristics are to be found.

VII. INTERPLANETARY TRAJECTORIES

As an example of the application of this theory to mission feasibility studies, the results from an extensive set of rendezvous trajectories from Earth to Mars are presented. D. E. Richardson (JPL) has programmed the spherical coordinate formulation of these equations for numerical solution on an IBM 7090 digital computer. In order to overcome the two-point boundary-value problem, this program has been coupled with an iterative routine to converge upon the specified terminal conditions. By this procedure, parametric analyses have been efficiently conducted through the large-scale production of interplanetary trajectories with wide ranges of mission conditions and flight times.

In these examples, a two-dimensional inverse square force field model is employed. The departing orbit is circular and one astronomical unit from the Sun. The arrival orbit has the semi-major axis and the eccentricity of the Martian orbit. The terminal conditions are, therefore, specified values of energy and angular momentum; the transversality condition of $M(t_1) = 0$ is specified (except in Figs. 8 and 9) and, in all trajectories, the polar angle $\theta(t_1)$ is unspecified and, consequently, K_1 is zero. Both the variable and the constant thrust programs have been used; unless otherwise noted, the exhaust velocity employed in the constant thrust program in 50,000 m per sec.

Figure 1 exhibits a 160-day rendezvous trajectory to Mars using the variable thrust program. The arrows indicate both the direction and the magnitude of the thrust acceleration. Figures 2 and 3 show constant thrust trajectories accomplishing the same mission using an optimum coast period and no coast period, respectively. The coast period shown in Fig. 2 has been optimized so that $\int_0^{t_1} a^2\,dt$ is a minimum, $[R(t_1) = 0]$. Figures 4, 5, and 6 show the thrust programs for these three trajectories. The quantity γ is the angle between the thrust vector and the radius vector and is therefore given by

$$\tan\gamma = \frac{\lambda_2}{\lambda_1} \tag{94}$$

The switching function for the optimum coast trajectory is also shown in Fig. 5.

These three trajectory types have been run for different flight times ranging from 40 days to more than 300 days. Figure 7 shows the variation of $\int_0^{t_1} a^2\,dt$ with flight time for rendezvous at both the optimum and the worst point on the orbit of Mars. For continuous thrust interplanetary trajectories, it is known (2,9) that the effect of the planetary orbital inclinations on the value of $\int_0^{t_1} a^2\,dt$ or final vehicle mass is quite small; in the case of Mars, the 1.°85 orbital inclination increases the value of $\int a^2\,dt$ in three-dimensional variable thrust trajectories by less than 1%. Consequently, Fig. 7 yields a highly valid estimate of the payload capabilities of power-limited systems for Mars rendezvous missions. As stated previously, the $\int a^2\,dt$ is fairly insensitive to the exhaust velocity employed in typical cases, but it does become sensitive for extremely low values or short flight times. The following tabulation compares this integral for exhaust velocities of 50,000 meters per sec and infinity (constant acceleration) for Mars optimum coast trajectories.

T days	$\int a^2\,dt$	
	$C = 50{,}000$	$C = \infty$
50	556.03	525.11
100	59.821	59.250
150	15.263	15.225
200	5.6225	5.6138
250	2.7279	2.7222
300	1.1380	1.1338

Figures 8 and 9 show, for a 160-day flight, the variation of $\int a^2\,dt$ and $M(t_1)$ with true anomaly η on the Mars orbit for the optimum-coast and minimum-time programs. The behavior of these quantities for the variable thrust program is similar and is discussed in Reference 2. The values of true anomaly where $M(t_1) = 0$ depend on the flight time (2,9), but, generally, the minimum point lies in the first and second quadrant.

In Fig. 10, the variation of $\int a^2\,dt$ and the corresponding coast period with initial thrust acceleration is shown. The transversality condition $R(t_1)$ is also plotted, and its zero crossing marks the minimum $\int_0^{t_1} a^2\,dt$. An initial acceleration of 0.0009008 m per sec^2 is the smallest value that may be used to accomplish this mission and yields, of course, the minimum-time trajectory. The abrupt drop in $\int a^2\,dt$, with a small increase in a_0, suggests the value of employing trajectories with a coast period.

Many of the curves presented here can be approximately reproduced using the simple one-dimensional analysis in Sec. VI. As examples, from Equations (88) and (89), one may obtain the expression

$$\int_0^{t_1} a^2\, dt = \frac{4a_0 L}{\left(t_1 + \sqrt{t_1^2 - \frac{L}{a_0}}\right)} \tag{95}$$

which, when plotted against a_0, bears a striking resemblance to the analogous curve in Fig. 10. From the same equations, one obtains the coast time

$$t_c = t_1 - \frac{4L}{a_0} \tag{96}$$

which is also similar to the t_c curve in Fig. 10. Furthermore, the $\int a^2\, dt$ follows very closely a t_1^{-3} variation with flight time, and, for a given mission, the $\int_0^{t_1} a^2\, dt$ values for the variable thrust program, optimum-coast program and minimum-time programs are approximately in the ratio 1:1.13:1.33, as obtained in the one-dimensional analysis.

Finally, Fig. 11 exhibits a case in which the criterion of maximum final vehicle mass is employed. In this set of trajectories, the exhaust power is fixed and various values of the exhaust velocity are used to accomplish this mission. The resulting curves for μ, λ_{11} and t_c are presented. As before, the zero crossing of λ_{11} marks the maximum value of μ; therefore, setting λ_{11} equal to zero as a terminal condition yields the optimum exhaust velocity to be employed for this mission.

If payload maximization is the desired end, the procedure described in Fig. 11 is strictly the correct approach; however, these curves are strongly dependent on the value of β employed and, consequently, this procedure is dependent on the power and efficiency ratings of the particular propulsion system under consideration. For parametric studies, the $\int_0^{t_1} a^2\, dt$, because of its near invariance to the propulsion system ratings, is of more utility.

NOMENCLATURE

A	constant of integration
A_ν	initial boundary functions
$\mathbf{a}$	thrust acceleration vector
B_ν	terminal boundary functions
c	exhaust velocity
E	total energy per unit mass
e	eccentricity
$\mathbf{e}_\gamma, \mathbf{e}_\theta, \mathbf{e}_\phi$	basis vectors in spherical coordinate system
F	generalized integrand
$F(t)$	spherical coordinate Lagrange multiplier
G_i	constraining functions
GM	gravitational coefficient of central body
h	angular momentum per unit mass
i	inclination
J	generalized function to be extremized
K_1	constant of integration
K_2	constant of first integral of Euler equations
L	distance (Sec. VI)
$L(t)$	switching function
m	vehicle mass
M	transversality function
N	transversality function
P	exhaust power
q_j	generalized coordinates

NOMENCLATURE (Cont'd)

$\mathbf{r}$	position vector
r	radial distance
$R(t)$	transversality function
t	trajectory time
t_1	flight time
t_a	time of commencement of coast period
t_b	time of commencement of second burn period
U	potential function
$\mathbf{v}$	velocity vector
α_p	normalized power parameter
β	twice the ratio of maximum exhaust power to initial vehicle mass
γ	angle between thrust vector and radius vector, (Equation 94)
γ	auxiliary real variable
η	auxiliary real variable
η	true anomaly (Sec. VII)
θ	polar angle
κ_ν	initial boundary Lagrange multiplier
λ_i	Lagrange multipliers
μ	normalized vehicle mass
ρ_ν	terminal boundary Lagrange multipliers
ϕ	latitude
ψ	position angle of ascending line of nodes
ω	argument of perigee
Ω	longitude of ascending node

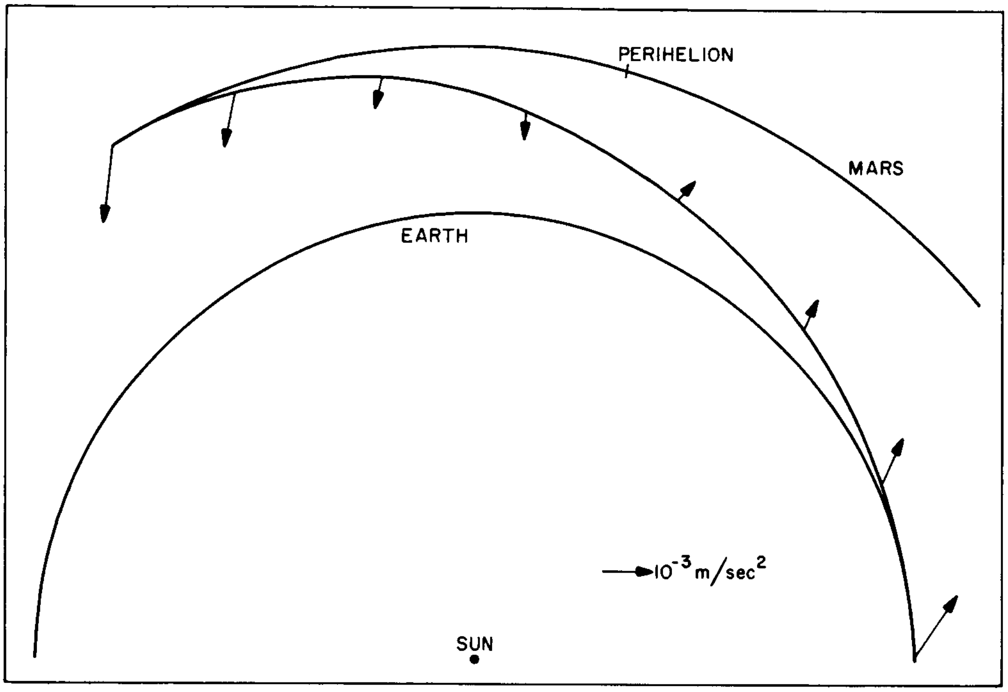


Fig. 1. Mars optimum rendezvous variable thrust trajectory; 160-day flight time

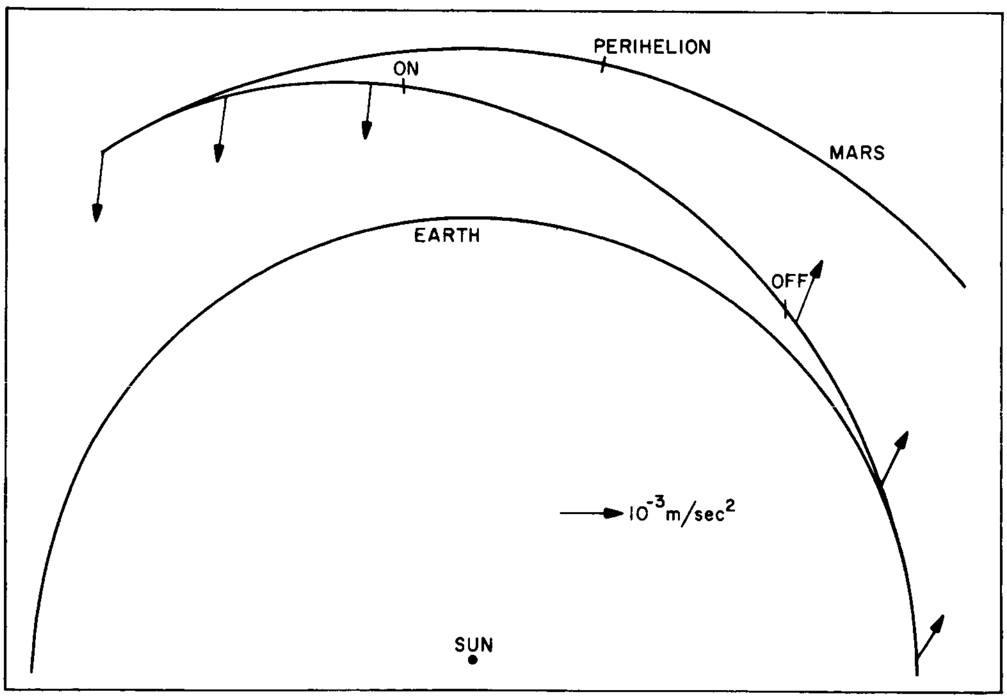


Fig. 2. Mars optimum rendezvous constant thrust trajectory; 160-day flight time, optimum coast; exhaust velocity = 50,000 m/sec

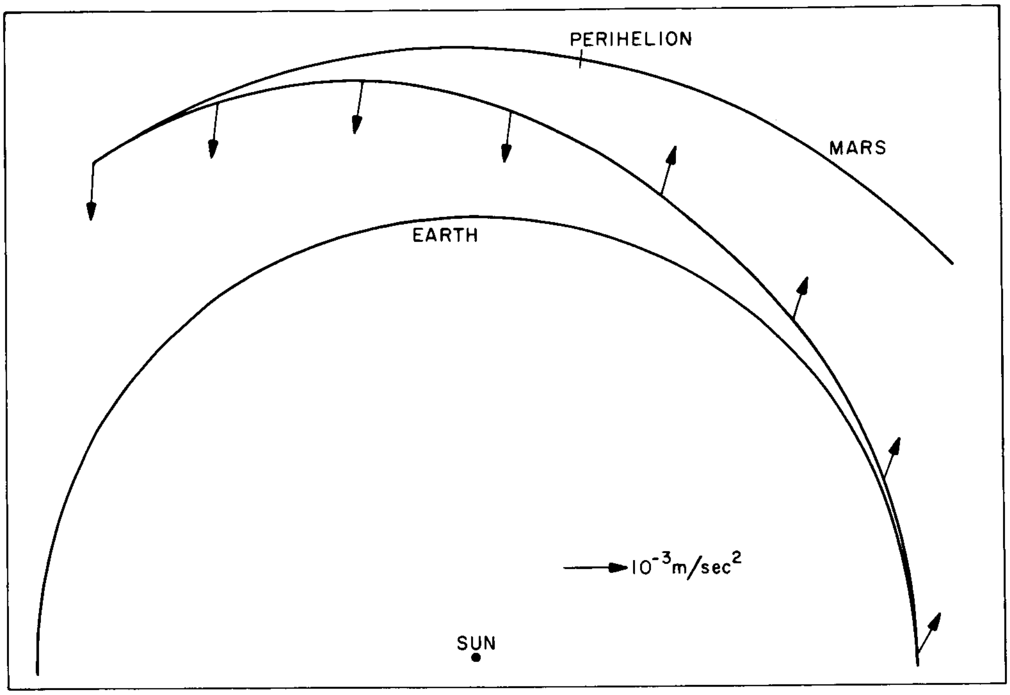


Fig. 3. Mars optimum rendezvous constant thrust trajectory; 160-day flight time, minimum time trajectory; exhaust velocity = 50,000 m/sec

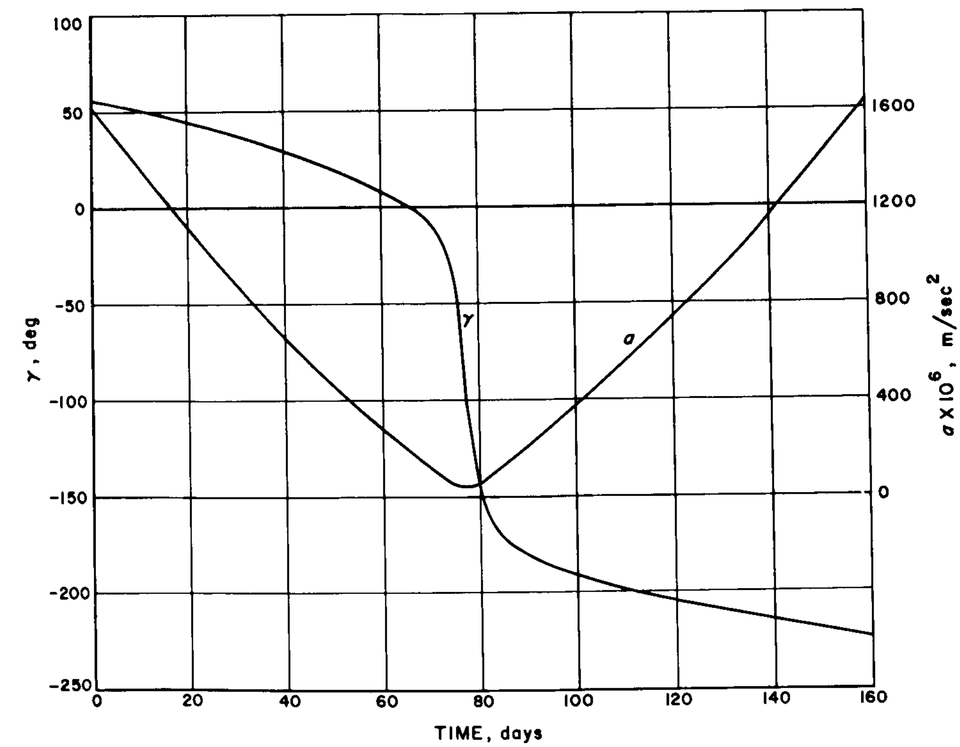


Fig. 4. Thrust program for variable thrust trajectory

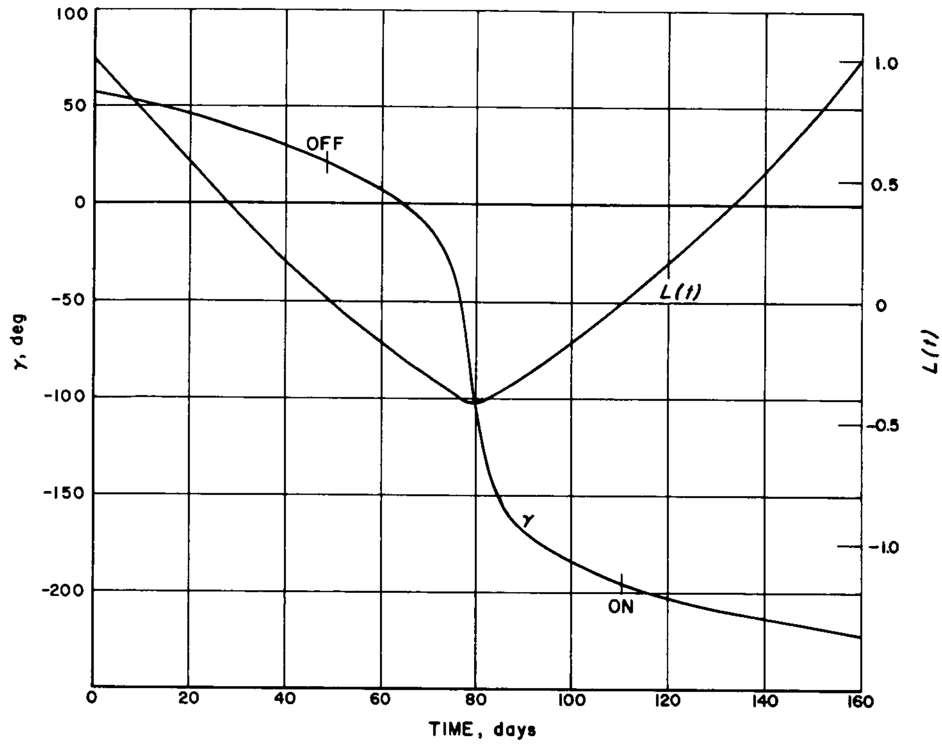


Fig. 5. Thrust program and switching function for optimum coast trajectory

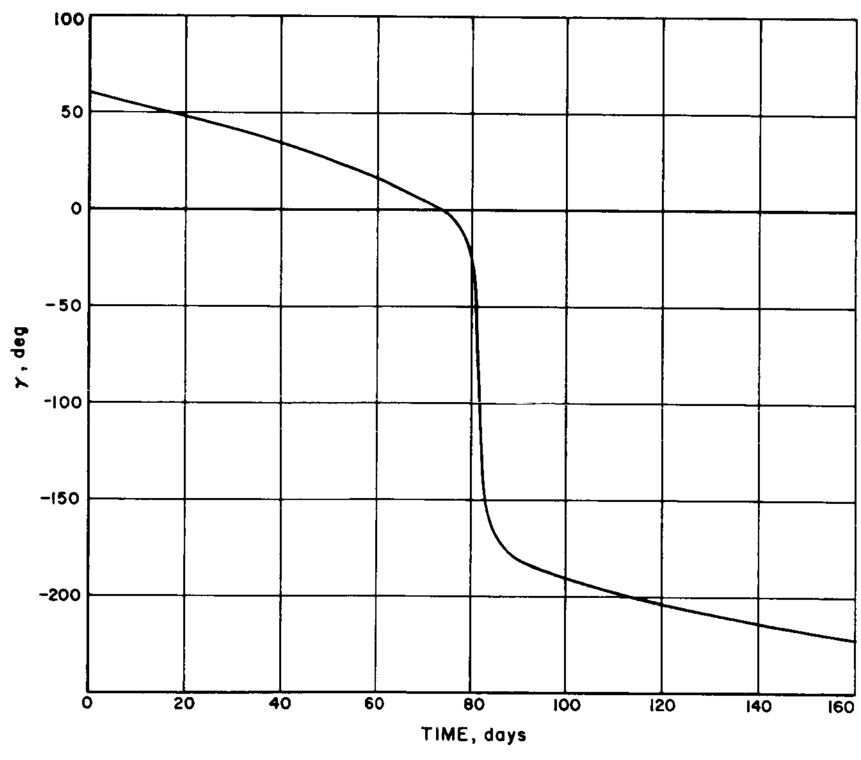


Fig. 6. Thrust program for minimum time trajectory

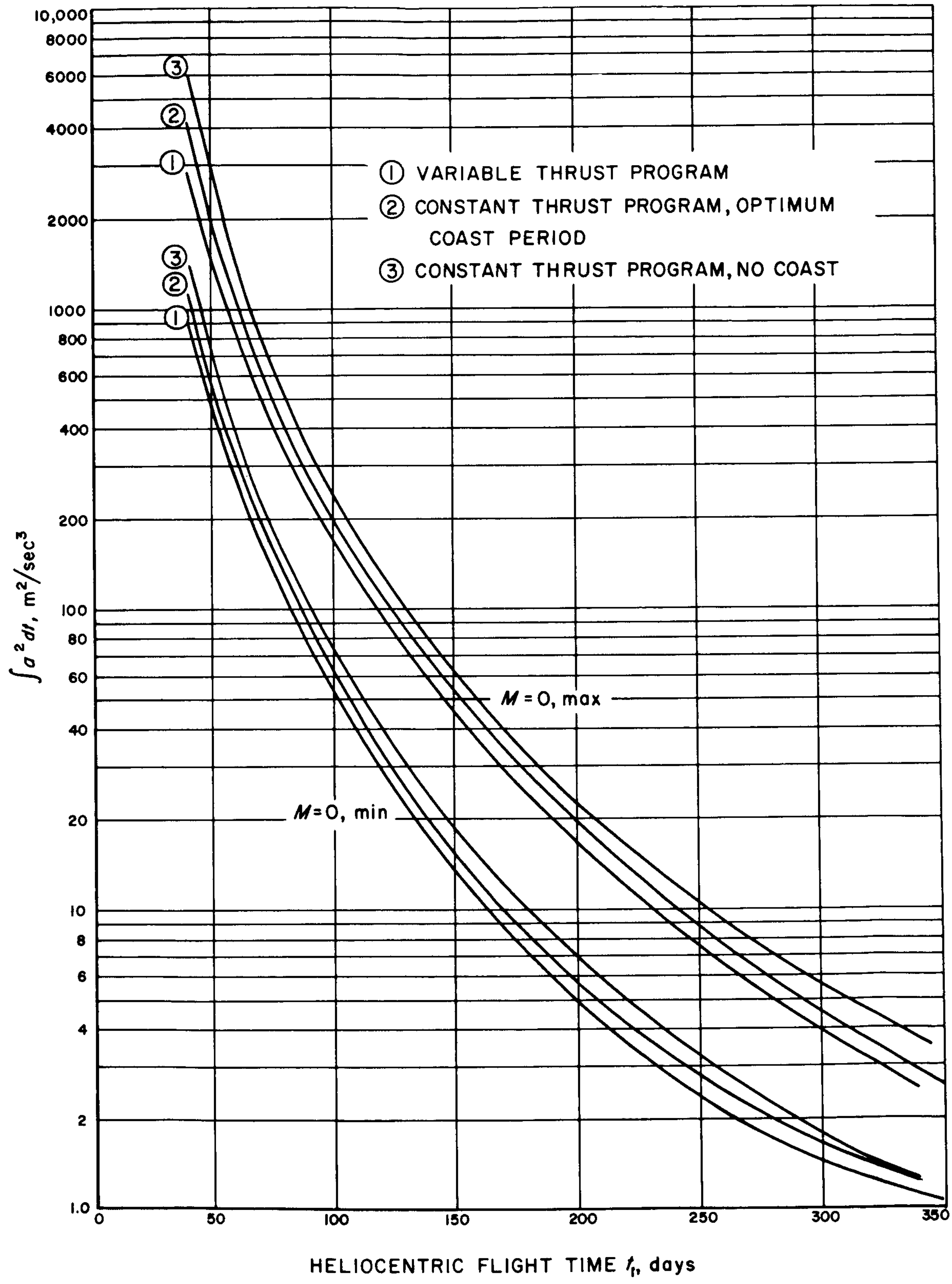


Fig. 7. Mars rendezvous trajectories; variation of $\int_0^{t_1} a^2\,dt$ with flight time

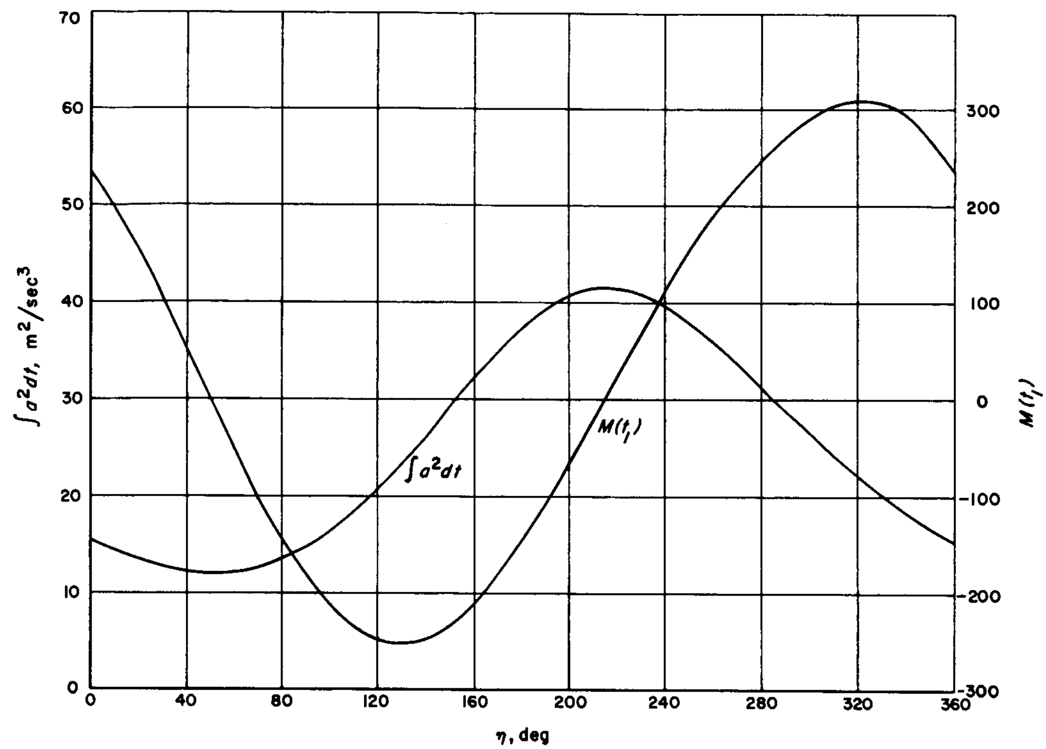


Fig. 8. Mars 160-day constant thrust optimum coast trajectories, variation of $\int_0^{t_1} a^2\, dt$ with rendezvous point on Martian orbit

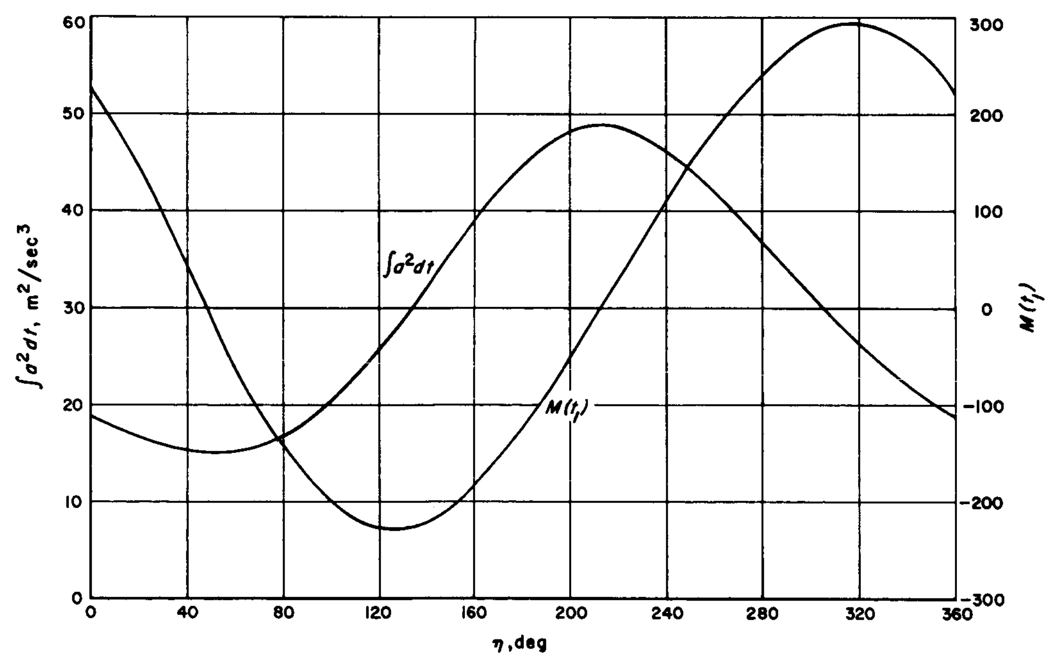


Fig. 9. Mars 160-day constant thrust minimum time trajectories; variation of $\int_0^{t_1} a^2\, dt$ with rendezvous point of Martian orbit

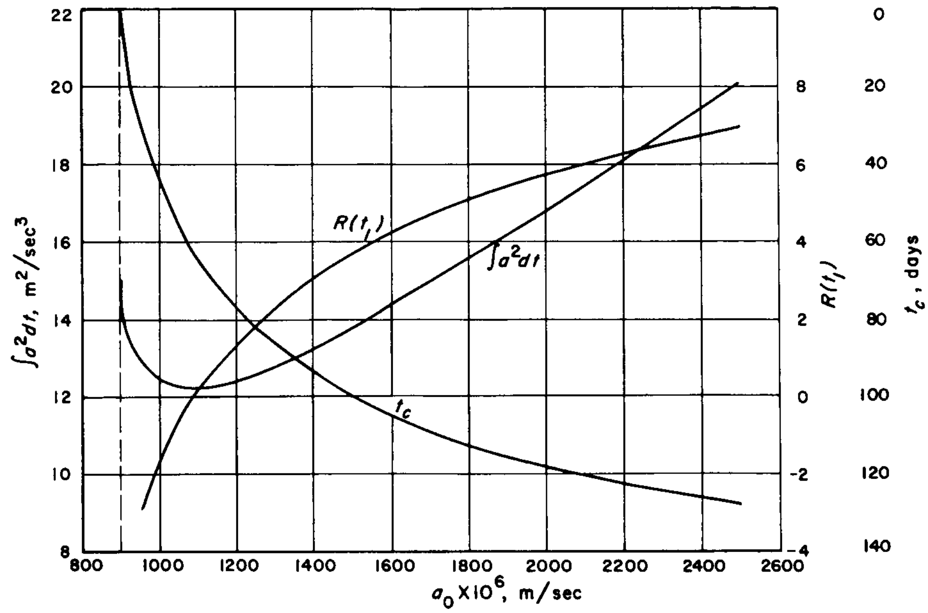


Fig. 10. Mars 160-day constant thrust trajectories; variation of $\int a^2\,dt$, $R(t)$, and t_c coast period with initial thrust acceleration; exhaust velocity = 50,000 m/sec

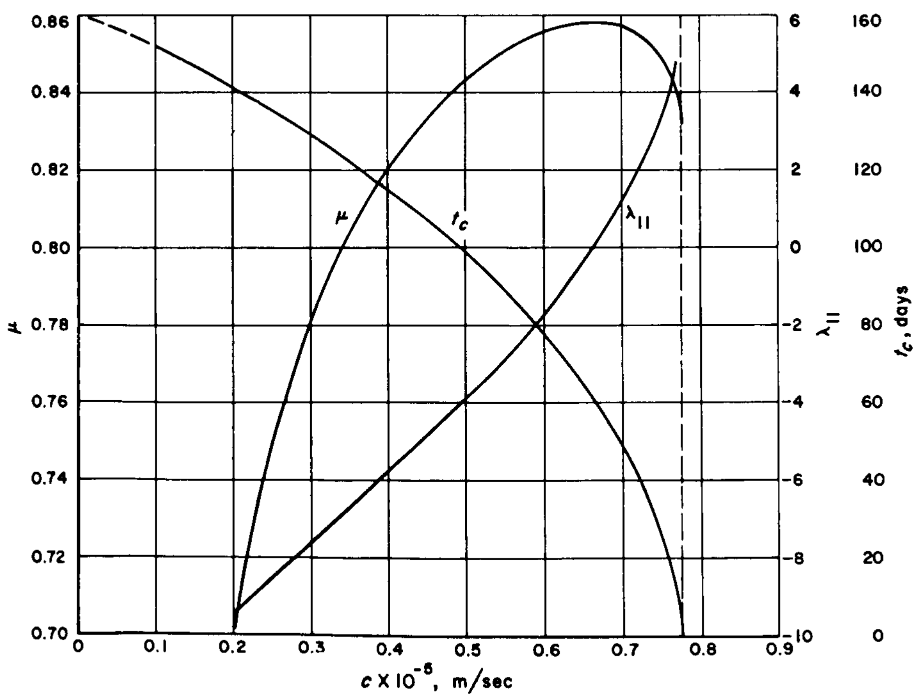


Fig. 11. Mars 160-day constant thrust trajectories; variation of μ, λ_{11}, and t_c coast period with exhaust velocity; $\beta = 73.788$ m^2/sec^3

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APPENDIX

A Discretely Varying Thrust Program

Suppose the exhaust velocity is constrained to discrete values

$$c = c_1 = c_{min},\ c_2,\ c_3,\ \cdots,\ c_r = c_{max} \tag{A-1}$$

The Euler-Lagrange equations, etcetera, all still apply. However, it remains to determine the conditions for changing the exhaust velocity discontinuously. The continuity of K_2 requires that $a_p L/c$ is continuous. Let c_- and c_+ be the exhaust velocities on the left-and right-hand side of a discontinuity, respectively, and thus

$$\begin{aligned} c_- &= c_i \\ c_+ &= c_{i-1},\ \text{or}\ c_{i+1} \end{aligned} \tag{A-2}$$

Equation (12) requires that

$$\frac{L_+}{c_+} = \frac{L_-}{c_-} \tag{A-3}$$

and using Equation (25), the condition on L_- for a transition in c is

$$L_- = \frac{c_-}{c_- + c_+}\,\frac{\lambda}{\mu} \tag{A-4}$$

and, on the right-hand side of the discontinuity, L becomes

$$L_+ = \frac{c_+}{c_- + c_+}\,\frac{\lambda}{\mu} \tag{A-5}$$

In both cases, L is positive and α_p is, therefore, unity, as has been implicitly assumed. Furthermore, these conditions show that L cannot become negative until c_{max} is reached.

Equations (A-4) and (A-5) may be used to prove the continuity in c when changing thrust modes in the variable- and constant-thrust programs.